Let $\mathbf{v} = (v_1, v_2, \ldots, v_n) > 0$ be an n-dimensional vector of values, $\mathbf{e} = (1, 1, \ldots, 1)$ the n dimensional vector of all ones, $\mathbf{a} = \mathbf{v}/(\mathbf{e}^T\mathbf{v})$ the corresponding target vector, and $\mathbf{x}^0$ a starting vector with the property that $\mathbf{e}^T\mathbf{x}^0 = 1$. We seek a procedure that generates a sequence of vectors $\mathbf{x}^0, \mathbf{x}^1, \ldots$ that converges to $\mathbf{a}$. Notice that a vector correponds to a point in the nth dimensional Euclidean space, and so we define the Euclidean distance $d(\mathbf{v}^1, \mathbf{v}^2)$ between two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ as the Euclidean distance between the two corresponding points. In particular, we will be interested in the rate of convergence to zero of $d(\mathbf{x}^t, \mathbf{a})$. The ith component of $\mathbf{x}^t$ will be denoted as $\mathbf{x}_i^t$. At each stage, we are allowed to choose two components i and $j > i$ and redistribute their value, that is, $\mathbf{x}^{t+1} = (x_1^t, \ldots, x_i^{t+1}, \ldots, x_j^{t+1}, \ldots, x_n^t)$, where $x_i^{t+1} + x_j^{t+1} = x_i^t + x_j^t \stackrel{\text{def}}{=} s_t$. We propose the algorithm that assigns

$$x_i^{t+1} = s_t \frac{v_i}{v_i + v_j} . \tag{1}$$

Lemma 1 *When $n = 3$, algorithm (1) yields*

$$E[d(\mathbf{x}^{t+1}, a)|\mathbf{x}^t, \mathbf{x}^{t-1} \ldots, \mathbf{x}^0] \leq \frac{4}{3\sqrt{3}} E[d(\mathbf{x}^t, \mathbf{a})|\mathbf{x}^{t-1}, \ldots, \mathbf{x}^0] ,$$

where the expectations is taken over random choices of i and j.

Proof. The proof is a geometric construction of $\mathbf{x}^{t+1}$ on the basis of $\mathbf{x}^t$, which yields an upper bound on $d(\mathbf{x}^t, \mathbf{a})$ that does not depend on previous $\mathbf{x}$'s. First, we define the set of feasible vector and show that it is a triangle. Moreover, we will show how to construct $\mathbf{x}^{t+1}$ from $\mathbf{x}^t$ through an elementary geometric projection. A trigonometric analysis of such projection will yield the desired bound.

Define the set of feasible points as the triangle $x_1 + x_2 + x_3 = 1$, $x_1, x_2, x_3 \geq 0$. Observe that such triangle is equilateral and so its angles are $\pi/3$. Suppose first that $i = 1$, $j = 2$, and so x_3 is left unchanged in the current step. Hence, the resulting vector will be a point in the segment defined by $x_1 + x_2 + x_3 = 1$, $x_1, x_2, x_3 \geq 0$, and $x_3 = x_3^t$. Notice that the line defined by $x_1 + x_2 + x_3 = 1$ and $x_3 = x_3^t$ is parallel to the side of triangle with $x_3 = 0$. In general, a step moves the current vector $\mathbf{x}^t$ to a new point $\mathbf{x}^{t+1}$ along a line which is parallel to a triangle side. Hence, there are only three lines along which movement can occur. Let α be the angle between any two of those lines, and observe that $\alpha = \pi/3$ by an elementary property of parallel lines which is depicted in figure 1. Let u, w, and z be the three lines parallel to triangle sides passing through $\mathbf{x}^t$, and notice that these three lines partition the plane into size equal angles. We next give simple geometric construction to determine $\mathbf{x}^{t+1}$ assuming that the coordinate x_3 is left unchanged. Consider first the triangle ACD in figure 2 and notice that such triangle is rectangle and its base is s_t, which is equal to its height. Analogously, EGH is rectangle and both its base and height are $a_2 + a_3$. Hence, ACD and EGH are similar. Let us turn now to AB$\mathbf{x}^{t+1}$ and notice that it is rectangle, its base is x_1^{t+1}, and its height is x_2^{t+1}. Hence, height and base are in the ratio v_1/v_2. However, the triangle EG$\mathbf{a}$ is also rectangle and its base and height are in the same ratio v_1/v_2. Hence, AB$\mathbf{x}^{t+1}$ and EG$\mathbf{a}$ are similar. Therefore, we can use the following construction to determine $\mathbf{x}^{t+1}$. Suppose that $\mathbf{x}^{t+1}$ is chosen along line w, as in figure 1. Take a line l that passes through

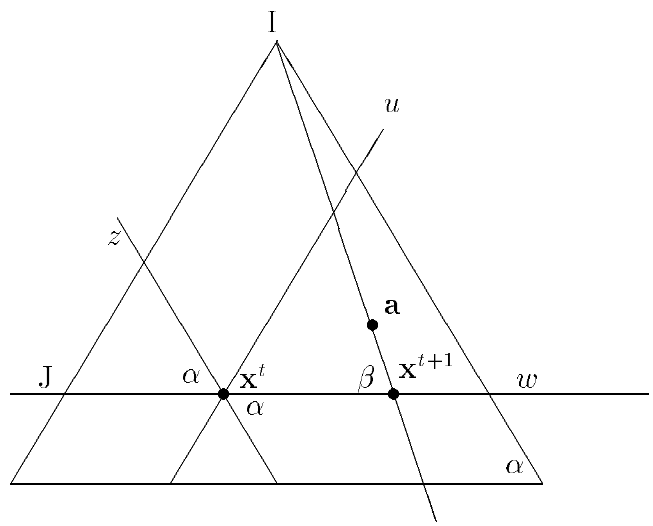

Figure 1: Geometric construction of $\mathbf{x}^t$.

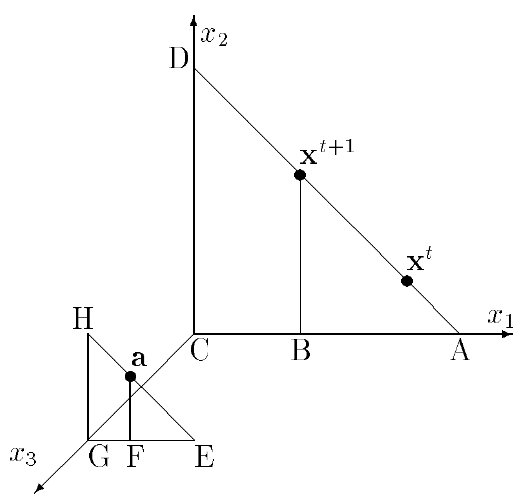

Figure 2: Similarity between triangles.

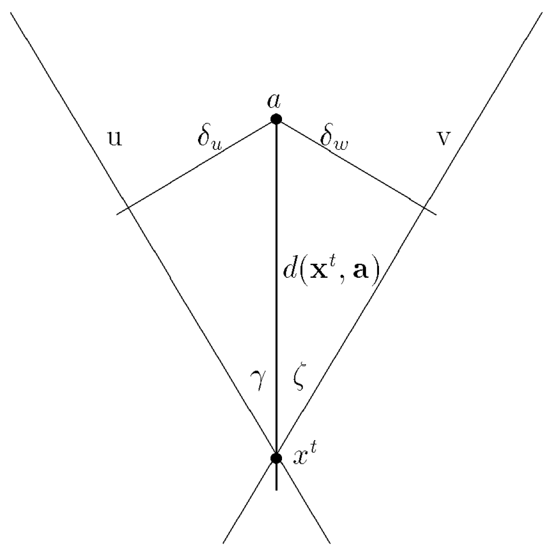

Figure 3: Relation between distances.

the vertex I opposite to w and through $\mathbf{a}$. Then, $\mathbf{x}^{t+1}$ is located at the intersection of l and w.

We next turn to show a bound on the angle β formed by l and w, and specifically that $\pi/3 < \beta < 2\pi/3$. Indeed, take the line passing through $\mathbf{x}^{t+1}$ that is parallel to z, and observe that $\beta \geq \alpha = \pi/3$. As for the second part of the statement, notice that the angle J is $\pi/3$, and so by the sum of the angles in the triangle IJ$\mathbf{x}^{t+1}$, $\beta < 2\pi/3$. Assume without loss of generality that $\mathbf{a}$ is in the partition determined by u and w, and denote by δ_w (δ_u, δ_z) the distance between $\mathbf{a}$ and w (u, z). We will show that the δ's are not much smaller that $d(\mathbf{x}^{t+1}, \mathbf{a})$. Specifically, if $\mathbf{x}$ is chosen along w, then $d(\mathbf{x}^{t+1}, \mathbf{a}) = \delta_w \sin\beta \leq 2\delta_w/\sqrt{3}$ because $\pi/3 \leq \beta < 2\pi/3$. Analogous results hold also for the other lines. We now turn to estimate an upper bound to the expected value of δ for a random choice of i and j. First, notice that $\delta_z \leq d(\mathbf{x}^t, \mathbf{a})$ by definition of δ_z. Moreover, the line $\mathbf{x}^t\mathbf{a}$ partitions a $\pi/3$ angle into angles γ and ζ with $\gamma + \zeta = \pi/3$ as in figure 3. Hence, $\delta_u = d(\mathbf{x}^t, a)\sin\gamma$, $\delta_w = d(\mathbf{x}^t, a)\sin\zeta = d(\mathbf{x}^t, a)\sin(\pi/3 - \gamma)$. Hence, $\delta_u + \delta_w = d(\mathbf{x}^t, \mathbf{a})\sin(\gamma + \pi/3) \leq d(\mathbf{x}^t, \mathbf{a})$. On the whole, the expected δ is $(\delta_u + \delta_w + \delta_z)/3 \leq 2d(\mathbf{x}^t, \mathbf{a})$, so that the expected $d(\mathbf{x}^{t+1}, \mathbf{a}) \leq 4/3\sqrt{3}d(\mathbf{x}^t, \mathbf{a})$. We notice that such bound is independent of the previous values of $\mathbf{x}^t, \mathbf{x}^{t-1}, \ldots, \mathbf{x}^0$, and the lemma is proven. □

The repeated application of the previous lemma yields that $\mathbf{x}^t$ converges expoinentially

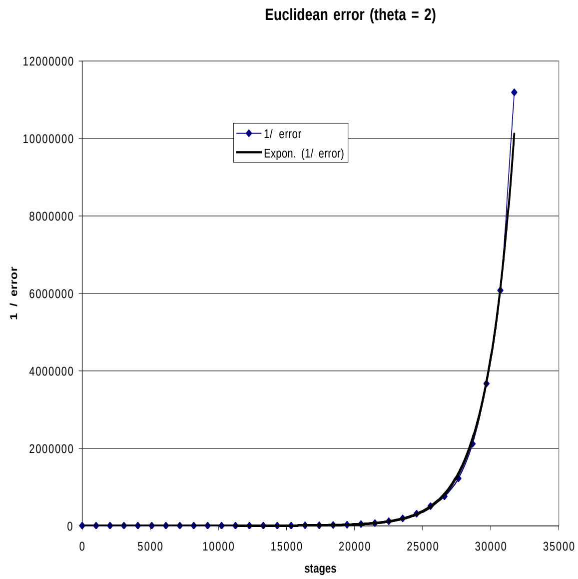

Figure 4: $n = 1024$ for $\theta = 2$.

fast to **a**.

Our experimental results suggest that the proposition generalizes when $n > 3$, as shown in figure 4. The convergence has is given for $n = 1024$, where $v_i = i^{-\theta}$ for $\theta = 2$.